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ENERGY RELEASE RATE AND CRACK KINKING

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Abstract—A closed form solution is presented for the energy release rate at the onset of kinking of a straight crack in an infinite elastic medium subjected to a predominantly Mode I loading. The solution is accurate to the second order of kink angle and is carried out by the method which models the kink as a continuous distribution of infinitesimal edge dislocations. On the basis of the maximum energy release rate criterion, simple expressions are obtained for the critical kink angle and the critical applied stress. The examination of the results shows that the second order solutions are in very good agreement, up to fairly large kink angles, with numerical results reported by others. It is also analytically established that the Irwin formula for the energy release rate remains valid under the predominantly Mode I loading, provided that the stress intensity factors in the formula are appropriately interpreted.

1. INTRODUCTION

In accordance with concepts in brittle fracture, much effort has been devoted by a number of workers to the calculation of the stress intensity factors for kinked cracks and, recently, a convincing solution has been presented in [1], in which the crack is modeled as a continuous distribution of infinitesimal edge dislocations. In addition to the knowledge of the stress intensity factors, a fracture criterion must first be established from physical laws in order to analyze the phenomenon of crack kinking. There are currently three fracture criteria available for this purpose, namely, the maximum hoop stress criterion[2], the minimum strain energy density criterion[3] and the maximum energy release rate criterion which is a generalization of Griffith's original energy release rate criterion[4, 5]. The first two criteria stand on their own merits, whereas the last one seems to stem from the fundamental mechanic principle of minimum potential energy. In recent years, several analytical studies have appeared, which concern the so-called mixed-mode fracture and employ the maximum energy release rate criterion[6–11]; in particular, the results in[9–11] appear to be rather complete. Except for [6], all results so far obtained are numerical ones.

The main objective of the present paper is to study the energy release rate at the onset of kinking of a straight crack under a predominantly Mode I loading. To this end, the problem of a kinked crack in an infinitely extended medium is first analyzed by the method which models the kink as a continuous distribution of infinitesimal edge dislocations. In the analysis, the complex potential functions[12] satisfying the traction-free conditions on the main crack, are employed, similarly to [1, 13]. Therefore, the resulting integral equations for the dislocation density functions are defined on the line of the kink only, and an analytical solution, accurate to the second order of kink angle, is obtained. Since the density functions correspond to the derivatives of the displacement-discontinuities across the kink with respect to the distance from the end of the kink, the energy release rate is easily obtained with the aid of the tractions that prior to the onset of kinking act on the line of the kink.

Closed form expressions for the stress intensity factors and the energy release rate at the onset of kinking are obtained for small kink angles. These expressions indicate that the Irwin formula[14] for the energy release rate remains valid at the onset of kinking under the predominantly Mode I loading, provided that the stress intensity factors in this formula are interpreted as those corresponding to a kink of almost zero length. On the basis of the maximum energy release rate criterion, simple expressions are also obtained for the critical kink angle and the critical applied stress. The examination of the results shows that these simple analytically obtained expressions agree very well with the numerical results presented in [11], up to fairly large kink angles.

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2. STATEMENT OF PROBLEM

We consider the plane strain problem of a kinked crack in an infinite elastic medium. Far from the crack, the medium is subjected to a uniform tensile stress which is nearly perpendicular to the main crack, so that if crack kinking occurs, the kink angle ω, would be sufficiently small. The fized rectangular Cartesian coordinate system x_α, shown in Fig. 1, is used; throughout this work, Greek indices take on the values 1, 2, unless otherwise stated. We also use a *supplementary* rectangular Cartesian coordinate system, ζ_α, as shown in Fig. 1. The relation between the two coordinate systems is given by

$$\zeta = z\,e^{-i\omega}, \tag{1}$$

where $z = x_1 + ix_2$, $\zeta = \zeta_1 + i\zeta_2$. In what follows, the superscript 0 is used to identify functions in the supplementary coordinate system.

With the aid of the elastic potential functions$\Phi(z)$, $\Psi(z)$[12], which are holomorphic in the region occupied by the medium, we express the boundary conditions of the problem as follows:

(a) On the surface of the main crack L, $\sigma_{22} - i\sigma_{12} = 0$:

$$\Phi(x_1) + \overline{\Phi(x_1)} + x_1\overline{\Phi'(x_1)} + \overline{\Psi(x_1)} = 0 \quad \text{for } x_1 \in L. \tag{2}$$

(b) On the surface of the kink L',$\sigma^0_{22} - i\sigma^0_{12} = 0$:

$$\Phi^0(\zeta_1) + \overline{\Phi^0(\zeta_1)} + \zeta_1\overline{\Phi^{0\prime}(\zeta_1)} + \overline{\Psi^0(\zeta_1)} = 0 \quad \text{for } \zeta_1 \in L'. \tag{3}$$

(c) Far from the crack:

$$\Phi(z) = \frac{\sigma^\infty_{11} + \sigma^\infty_{22}}{4} + 0\left(\frac{1}{z^2}\right), \quad \Psi(z) = \frac{\sigma^\infty_{22} - \sigma^\infty_{11}}{2} + i\sigma^\infty_{12} + \; + 0\left(\frac{1}{z^2}\right) \text{ as } |z| \to \infty. \tag{4}$$

Here, $\sigma_{\alpha\beta}$ denotes the stress component referred to the x_α-coordinates,

$$\sigma^\infty_{11} = \sigma^\infty \sin^2\gamma, \; \sigma^\infty_{22} = \sigma^\infty \cos^2\gamma, \; \sigma^\infty_{12} = \sigma^\infty \sin\gamma\cos\gamma, \tag{5}$$

and an overbar is used to indicate the complex conjugate.

In view of (1), $\Phi^0(\zeta)$ and $\Psi^0(\zeta)$ are given by

$$\Phi^0(\zeta) = \Phi(\zeta\,e^{i\omega}), \; \Psi^0(\zeta) = \Psi(\zeta\,e^{i\omega})\,e^{2i\omega}. \tag{6}$$

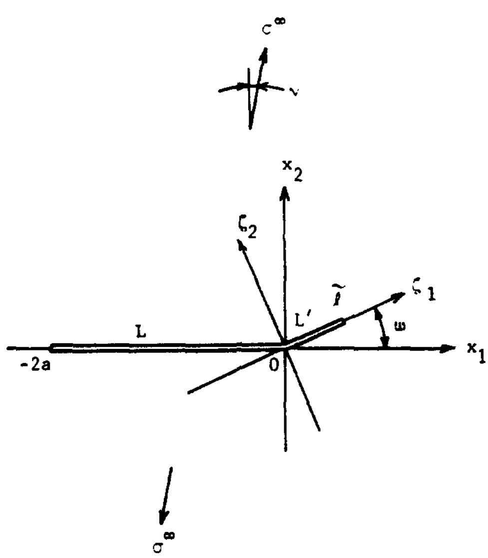

Fig. 1. Geometry and coordinate systems.

3. SINGULAR INTEGRAL EQUATIONS FOR SMALL KINK ANGLE

When an edge dislocation line perpendicular to the z-plane, passes through a point $z = \alpha (= s\,e^{i\omega})$ in an infinite medium which is subjected to the remote tension prescribed in the previous section and contains a traction-free crack located on $-2a < x_1 < 0$, then the corresponding elastic potentials $\Phi_1(z, \alpha)$ and $\Psi_1(z, \alpha)$ are found, in the absence of rigid body rotation, to be

$$\Phi_1(z, \alpha) = \frac{\sigma_{11}^\infty + \sigma_{22}^\infty}{4} + i\lambda \frac{\beta e^{i\omega}}{z-\alpha} + \Phi_1^*(z) + \Phi_2^*(z, \alpha), \tag{7}$$

$$\Psi_1(z, \alpha) = \frac{\sigma_{22}^\infty - \sigma_{11}^\infty}{2} + i\sigma_{12}^\infty - i\lambda \left[\frac{\bar{\beta}\, e^{-i\omega}}{z-\alpha} - \frac{\bar{\alpha}\beta\, e^{i\omega}}{(z-\alpha)^2}\right] + \Psi_1^*(z) + \Psi_2^*(z,\alpha),$$

where

$$\Phi_1^*(z) = \frac{\sigma_{22}^\infty - i\sigma_{12}^\infty}{2}[X(z)\,(z+a) - 1], \tag{8}$$

$$\Phi_2^*(z, \alpha) = -\frac{i\lambda}{2}\left[\beta\, e^{i\omega}\left\{\frac{1}{z-\alpha} + \frac{1}{z-\bar{\alpha}} - \frac{X(z)}{X(\alpha)}\cdot\frac{1}{z-\alpha} - \frac{X(z)}{X((\bar{\alpha})}\cdot\frac{1}{z-\bar{\alpha}}\right\}\right.$$

$$\left. + \bar{\beta}\, e^{-i\omega}(\bar{\alpha} - \alpha)\left\{\frac{1}{(z-\bar{\alpha})^2} - \frac{X(z)}{X(\bar{\alpha})}\cdot\frac{a+\bar{\alpha}}{\bar{\alpha}(\bar{\alpha}+2a)}\cdot\frac{1}{z-\bar{\alpha}} - \frac{X(z)}{X(\bar{\alpha})}\cdot\frac{1}{(z-\bar{\alpha})^2}\right\}\right], \tag{9}$$

$$\Psi_1^*(z) = -\Phi_1^*(z) + \overline{\Phi_1^*(\bar{z})} - z\Phi_1^{*\prime}(z), \tag{10}$$

$$\Psi_2^*(z, \alpha) = -\Phi_2^*(z) + \overline{\Phi_2^*(\bar{z}, \alpha)} - z\Phi_2^{*\prime}(z), \tag{11}$$

$$X(z) = 1/\sqrt{(z(z+2a))}, \tag{12}$$

$$\lambda = \frac{E}{8\pi(1-\nu^2)}, \tag{13}$$

$$\beta = b_1 + ib_2. \tag{14}$$

Here, E is Young's modulus and ν Poisson's ratio. The quantity b_α denotes the component of the Burgers vector in the direction of the ζ_α-axis. The branch of $X(z)$ is taken such that $X(z) \to 1/z$ as $|z| \to \infty$. These expressions are essentially equivalent to those in[1].

In order to formulate the original problem, edge dislocations of the kind mentioned above are distributed continously on the line of the kink. Consequently, the elastic potential functions, $\Phi(z)$ and $\Psi(z)$, which satisfy the boundary conditions (2) and (4), are expressed as

$$\Phi(z) = \frac{\sigma_{11}^\infty + \sigma_{22}^\infty}{4} + \Phi_1^*(z) + i\lambda \int_0^l \frac{\beta\, e^{i\omega}}{z - s\, e^{i\omega}}\, ds + \int_0^l \Phi_2^*(z, s\, e^{i\omega})\, ds,$$

$$\Psi(z) = \frac{\sigma_{22}^\infty - \sigma_{11}^\infty}{2} + i\sigma_{12}^\infty + \Psi_1^*(z) - i\lambda \int_0^l \left[\frac{\bar{\beta}\, e^{-i\omega}}{z - s\, e^{i\omega}} - \frac{s\beta}{(z - s\, e^{i\omega})^2}\right] ds$$

$$+ \int_0^l \Psi_2^*(z, s\, e^{i\omega})\, ds. \tag{15}$$

The density function $\beta(\zeta_1)$ is related to the displacement-discontinuities across the kink,

$$\beta(\zeta_1) = \frac{\partial}{\partial \zeta_1}([u_1^0] + i[u_2^0]), \tag{16}$$

where

$$[u_\alpha^0] = u_\alpha^0|_{\zeta_2\to 0^+} - u_\alpha^0|_{\zeta_2\to 0^-}. \qquad (17)$$

Here, u_α^0 denotes the displacement component referred to the ζ_α-coordinates.

It is readily verified that the displacement field determined from the above elastic potential functions satisfies the requirement for the single valuedness of the displacement on any closed circuit surrounding the kinked crack; see Appendix A.

Before setting up the corresponding singular integral equations, we introduce the following nondimentional notation:

$$\xi = \zeta_1/a,\ t = s/a,\ t_{\alpha\beta} = \sigma_{\alpha\beta}/\lambda,\ t^\infty_{\alpha\beta} = \sigma^\infty_{\alpha\beta}/\lambda,\ l = \bar{l}/a. \qquad (18)$$

In view of (6), the elastic potential functions referred to the ζ_α-coordinates, are obtained from (15). Substituting these potential functions into (3), we obtain a system of singular integral equations for the density functions. Since the integral equations have generalized Cauchy kernels of complicated form, it seems impossible to obtain closed form solutions. However, when l and ω are sufficiently small, we can reduce† these to the following very simple integral equations which are accurate to the second order in ω:

$$\int_0^l \frac{2\beta}{\xi - t}\,dt - \int_0^l \frac{1}{\sqrt{\xi}}\left[\frac{2\beta}{\sqrt{\xi}+\sqrt{t}} - \omega^2(\beta+\bar{\beta})\left\{\frac{1}{\sqrt{\xi}+\sqrt{t}} - \frac{3\sqrt{\xi}}{(\sqrt{\xi}+\sqrt{t})^2} + \frac{4\xi}{(\sqrt{\xi}+\sqrt{t})^3}\right\}\right]dt$$

$$+\frac{1}{i}\left[\frac{t^\infty_{22}}{\sqrt{(2\xi)}}C_{11} + \frac{t^\infty_{12}}{\sqrt{(2\xi)}}C_{12} + (t^\infty_{11} - t^\infty_{22})\omega^2 - i\left\{\frac{t^\infty_{22}}{\sqrt{(2\xi)}}C_{21} + \frac{t^\infty_{22}}{\sqrt{(2\xi)}}C_{22} - \frac{t^\infty_{11} - t^\infty_{22}}{2}2\omega\right\}\right] = 0,\ (0 < \xi < l), \qquad (19)$$

where $C_{\alpha\beta}$ are functions of the kink angle ω only, and are identical to those in the Irwin-Williams solution[15]. In the present case, they are given by

$$C_{11} = 1 - \frac{3}{8}\omega^2,\ C_{12} = -\frac{3}{2}\omega,\ C_{21} = \frac{1}{2}\omega,\ C_{22} = 1 - \frac{7}{8}\omega^2. \qquad (20)$$

4. ENERGY RELEASE RATE FOR SMALL KINK ANGLE

The quantity $(\beta + \bar{\beta})$ corresponds to the derivative with respect to ζ_1 of the discontinuity of the displacement in the ζ_1-direction and is of the order of ω in the present case.‡ Therefore, the second term in the second integral in (19) is of $O(\omega^3)$ which can be omitted. Under the condition state above, we obtain the required integral equations for $b_\alpha(t)$ as follows:

$$\int_0^l \frac{b_2(t)}{\xi - t}\,dt - \int_0^l \frac{b_2(t)}{\sqrt{\xi}(\sqrt{\xi}+\sqrt{t})}\,dt = \frac{\alpha_2}{2\sqrt{(2\pi\xi)}} - \frac{t^\infty_{22}}{2}\omega^2,$$

$$\int_0^l \frac{b_1(t)}{\xi - t}\,dt - \int_0^l \frac{b_1(t)}{\sqrt{\xi}(\sqrt{\xi}+\sqrt{t})}\,dt = \frac{\alpha_1}{2\sqrt{(2\pi\xi)}} + \frac{t^\infty_{22}}{2}\omega,\ (0 < \xi < l), \qquad (21)$$

where

$$\alpha_2 = t^\infty_{22}\sqrt{\pi}C_{11} + t^\infty_{12}\sqrt{\pi}C_{12},\ \alpha_1 = t^\infty_{22}\sqrt{\pi}C_{21} + t^\infty_{12}\sqrt{\pi}C_{22}. \qquad (22)$$

Equations (21) are easily integrated (Appendix B), whereupon the dislocation density functions

†By means of Taylor expansion with respect to ω.

‡Note that for a predominantly Mode I loading, it may be assumed that $\omega = O(\gamma)$ and hence $t^\infty_{12} = O(\omega)$ and $t^\infty_{11} = O(\omega^2)$. For this reason, C_{22} in (22) may be set equal to 1.

$b_\alpha(\xi)$ are

$$b_2(\xi) = -\frac{\alpha_2}{2\pi\sqrt{(2\pi)}}\frac{1}{\sqrt{(l-\xi)}} - \frac{t_{22}^{\infty}}{2\pi^2}\omega^2\left[\log\left|\frac{\sqrt{(l-\xi)}+\sqrt{l}}{\sqrt{(l-\xi)}-\sqrt{l}}\right| - 2\frac{\sqrt{l}}{\sqrt{(l-\xi)}}\right],$$

$$b_1(\xi) = -\frac{\alpha_1}{2\pi\sqrt{(2\pi)}}\frac{1}{\sqrt{(l-\xi)}} + \frac{t_{22}^{\infty}}{2\pi^2}\omega\left[\log\left|\frac{\sqrt{(l-\xi)}+\sqrt{l}}{\sqrt{(l-\xi)}-\sqrt{l}}\right| - 2\frac{\sqrt{l}}{\sqrt{(l-\xi)}}\right]. \tag{23}$$

From (23) it follows that the stress intensity factors at the tip of the kink are

$$K_I = \lambda\alpha_2\sqrt{a} - 2\sqrt{\frac{2}{\pi}}\sigma_{22}^{\infty}\omega^2\sqrt{\bar{l}} + 0(\omega^3), \tag{24}$$

$$K_{II} = \lambda\alpha_1\sqrt{a} + 2\sqrt{\frac{2}{\pi}}\sigma_{22}^{\infty}\omega\sqrt{\bar{l}} + 0(\omega^3),$$

where the stress intensity factors are defined as

$$K_I - iK_{II} = \lim_{\zeta_1\to\bar{l}^+}\sqrt{(2\pi(\xi_1-\bar{l}))}\,(\sigma_{22}^0 - i\sigma_{12}^0)|_{\zeta_2=0}. \tag{25}$$

In the limit as $\bar{l}\to 0$, the second term in (24) vanishes and the expressions reduce to those in [16]. It is stated in [16] that these expressions then are accurate to the first order in ω. Our results show that they are in fact accurate to the second order in ω under predominantly Mode I loading.

When a kink starts from a tip of a main crack, the change of the potential energy (ΔP) per unit thickness of the elastic body (plane strain) is given by

$$\Delta P = \frac{1}{2}\int_{\Delta S}\sigma_{\alpha\beta}^{*0}\,\Delta u_\alpha^0\,n_\beta^0\,\mathrm{d}S, \tag{26}$$

where ΔS denotes the newly created area by kinking, $\sigma_{\alpha\beta}^{*0}$ are the stresses acting on ΔS before the onset of kinking and referred to the supplementary coordinate system, Δu_α^0 are the displacement increments due to kinking, and $n_\alpha^0$is the outward unit normal vector of ΔS. From (26), we obtain the energy release rate, G, as follows:

$$G = -\frac{\partial(\Delta P)}{\partial\bar{l}}|_{\bar{l}\to 0} = \lim_{\bar{l}\to 0}\left[\frac{1}{2\bar{l}}\int_0^{\bar{l}}\Sigma_{\alpha 2}(\zeta_1)\left\{-\frac{\partial}{\partial\zeta_1}[u_\alpha^0]\right\}\mathrm{d}\zeta_1\right], \tag{27}$$

where

$$\Sigma_{\alpha 2}(\zeta_1) = \int^{\zeta_1}\sigma_{\alpha 2}^{*0}(\zeta_1)\mathrm{d}\zeta_1. \tag{28}$$

In deriving (27), we have used the following conditions:

$$\Sigma_{\alpha\beta}(0) = 0, \quad [u_\alpha^0]_{\zeta_1=\bar{l}} = 0. \tag{29}$$

Employing the Irwin–Williams expression for $\sigma_{\alpha\beta}^{*0}$, and using (16), (23) and (27), we obtain, accurate to the second order in ω,

$$G = \frac{1-\nu^2}{E}[K_{I0}^2 + K_{II0}^2], \tag{30}$$

or, alternatively,

$$G = \frac{1-\nu^2}{E}\left[k_1^2 + k_2^2 - 2k_1k_2\omega - \frac{1}{2}k_1^2\omega^2\right], \tag{31}$$

where

$$K_{I0} = \lambda\alpha_2\sqrt{a}, \qquad K_{II0} = \lambda\alpha_1\sqrt{a}; \tag{32}$$

$$k_1 = \sigma_{22}^{\infty}\sqrt{(\pi a)}, \qquad k_2 = \sigma_{12}^{\infty}\sqrt{(\pi a)}. \tag{33}$$

The expression (30) is formally the same as Irwin's expression, in which K_{I0} and K_{II0} are the stress intensity factors existing prior to the onset of kinking fracture. The validity of (30) for all kink angles is examined numerically in[11] and it is indicated that this relation does not hold under certain situations. In Table 1, the energy release rates calculated from (31) are compared with those given in Table 2 of[11]. The results obtained here are in very good agreement with those in[11].

Now, if it is assumed that the crack kinks along a path where the energy release rate is locally maximum, i.e. $\partial G/\partial\omega = 0$, it follows that the critical kink angle, ω_c, is

$$\omega_c = -2\frac{k_2}{k_1} = -2\gamma. \tag{34}$$

Furthermore, applying the Griffith energy balance, the critical applied stress is

$$\sigma_c^{\infty} = \sigma_{Ic}\left(1 - \frac{\gamma^2}{2}\right), \tag{35}$$

where σ_{Ic} denotes the critical stress in the pure Mode I loading. The critical kink angle given by (34) is in very good agreement with those presented in [7, 11] up to fairly large kink angles. For example, when $\gamma = 0.05\pi$, $\omega_c = -0.1\pi$ ($=18°$) according to the numerical results in [11], and this is exactly the same as that given by (34).

It is readily recognized from (24) and (31) that only the singular terms in the asymptotic expansion of the stresses acting in the vicinity of the main crack before kinking, determine the stress intensity factors and the energy release rate in the limit as the length of the kink becomes zero, and that the higher order terms in the expansion have no effect. This feature may also hold in more general cases than the present one, since the basic integral equations in such cases are formally of the same form as (19) and only the kernels of the second integral in (19) change.

5. CONCLUSIONS

An analytical solution, which is accurate to the second order in the kink angle, has been established for crack kinking under a predominantly Mode I loading. It is revealed that the Irwin formula for energy release rate holds true for the small kink angles under predominantly Mode I loading, provided that the stress intensity factors in the formula are interpreted as those

Table 1. Energy release rate, $G^* = G/[\pi\sigma^{\infty 2}a(1-\nu^2)/E]$

$(\frac{\pi}{2} - \gamma)/\pi$	ω/π	G^*	
		Wu [11]	Present Result
0.400	-0.176	1.061	1.073
0.417	-0.153	1.049	1.056
0.433	-0.133	1.035	1.040
0.450	-0.100	1.022	1.023
0.467	-0.067	1.010	1.011
0.483	-0.027	1.002	1.003
0.5	0	1.0	1.0

of the kinked crack with a kink of almost zero length. Furthermore, from the examination of the results, it appears that the simple expression for the critical kink angle obtained here, remains accurate for fairly large kink angles, approx. 20°.

Finally, it is shown that, at the onset of kinking, there are no constant term contributions to the energy release rate in the asymptotic expansion of the stress near the tip of a straight crack.

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REFERENCES

1. K. K. Lo, Analysis of branched cracks. *J. Appl. Mech.* **45**, 797–802 (1978).
2. F. Erdogan and G. C. Sih, On the crack extension in plates under plane loading and transverse shear. *J. Basic Engng* **85**, 519–527 (1963).
3. G. C. Sih, A special theory of crack propagation. *Mechanics of Fracture*, Vol. 1 Noordhoff, Leyden (1972).
4. A. A. Griffith, The phenomena of rupture and flow in solids. *Phil. Trans. R. Soc.* **A221**, 163–198 (1921).
5. A. A. Griffith, The theory of rupture. *Proc. 1st Int. Congr. Appl. Mech.*, 55–63 Delft (1924).
6. M. A. Hussain, S. L. Pu and J. Underwood, Strain energy release rate for a crack under combined Mode I and Mode II. *ASTM-STP-560*, 55–63 (1974).
7. K. Palaniswamy and W. G. Knauss, On the problem of crack extension in brittle solids under general loading. *Mechanics Today* (Edited by S. Nemat-Nasser) Vol. 4. pp. 87–148. Pergamon Press, Oxford (1978).
8. G. D. Gupta, Strain energy release rate for mixed mode crack problem. *ASME Paper No. 76–WA/PVP-7.*
9. C. H. Wu, Elasticity problems of a slender Z-crack. *J. Elasticity* **8**, 183–205 (1978).
10. C. H. Wu, Maximum-energy-release-rate criterion applied to a tension-compression specimen with crack. *J. Elasticity* **8**, 235–257 (1978).
11. C. H. Wu, Fracture under combined loads by maximum-energy-release-rate criterion. *J. Appl. Mech.* **45**, 553–558 (1978).
12. N. I. Muskhelishvili, *Some basic problems in the mathematical theory of elasticity.* Noordhoff, Leyden (1958).
13. B. L. Karihaloo, L. M. Keer and S. Nemat-Nasser, Crack kinking under nonsymmetric loading. To appear in *Engng Frac. Mech.*
14. G. R. Irwin, Analysis of stresses and strains near the end of a crack transversing a plate. *J. Appl. Mech.* **24**, 361–364 (1957).
15. M. L. Williams, On the stress distribution at the base of a stationary crack. *J. Appl. Mech.* **24**, 109–114 (1957).
16. B. Cotterell and J. R. Rice, Slightly curved or kinked cracks. *Int. J. Frac.* **16**, 155–169 (1980).
17. N. I. Muskhelishvili, *Singular Integral Equations.* Noordhoff, Leyden (1953).
18. C. H. Wu, Explicit asymptomatic solution for the maximum-energy-release-rate problem. *Int. J. Solids Structures* **15**, 561–566 (1979).

APPENDIX A

The requirement that the displacement is single-valued on any closed circuit surrounding the kinked crack is expressed by

$$\int_{-2a}^{0}\left\{\frac{\partial}{\partial x_1}(u_1^+ - u_1^-) + i\frac{\partial}{\partial x_1}(u_2^+ - u_2^-)\right\}\mathrm{d}x_1 + \mathrm{e}^{i\omega}\int_0^l \frac{\partial}{\partial \zeta_1}([u_1^0] + i[u_2^0])\,\mathrm{d}\zeta_1 = 0. \tag{A1}$$

Here and in the sequel, we denote the limiting values of a function as $x_2 \to 0^+$ and $x_2 \to 0^-$ by the superscript + and −, respectively.

From (15),

$$\frac{\partial}{\partial x_1}(u_1^+ - u_1^-) + i\frac{\partial}{\partial x_1}(u_2^+ - u_2^-) = \frac{8(1-\nu^2)}{E}\frac{\sigma_{22}^\infty - i\sigma_{12}^\infty}{2}X^+(x_1)(x_1 + a)$$

$$+\frac{i}{2\pi}\int_0^l\left[\beta\,\mathrm{e}^{i\omega}\left\{\frac{X^+(x_1)}{X(\alpha)}\frac{1}{x_1-\alpha} + \frac{X^+(x_1)}{X(\bar{\alpha})}\cdot\frac{1}{x_1-\bar{\alpha}}\right\} + \bar{\beta}\,\mathrm{e}^{-i\omega}\frac{X^+(x_1)}{X(\bar{\alpha})}\frac{\bar{\alpha}-\alpha}{x_1-\bar{\alpha}}\left\{\frac{a+\bar{\alpha}}{\bar{\alpha}(\bar{\alpha}+2a)} + \frac{1}{x_1-\bar{\alpha}}\right\}\right]\mathrm{d}s. \tag{A2}$$

The above equation, finally, leads to

$$\int_{-2a}^{0}\left\{\frac{\partial}{\partial x_1}(u_1^+ - u_1^-) + i\frac{\partial}{\partial x_1}(u_2^+ - u_2^-)\right\}\mathrm{d}x_1 = -\mathrm{e}^{i\omega}\int_0^l \beta\,\mathrm{d}s, \tag{A3}$$

where we have used

$$\int_{-2a}^{0}\frac{X^+(x_1)}{x_1-\alpha}\mathrm{d}x_1 = i\pi X(\alpha), \quad \int_{-2a}^{0}\frac{X^+(x_1)}{x_1-\bar{\alpha}}\mathrm{d}x_1 = i\pi X(\bar{\alpha}),$$

$$\int_{-2a}^{0}\frac{X^+(x_1)}{(x_1-\bar{\alpha})^2}\mathrm{d}x_1 = i\pi\left.\frac{\partial X}{\partial z}\right|_{z=\bar{\alpha}}, \quad \int_{-2a}^{0}X^+(x_1)(x_1+a)\mathrm{d}x_1 = 0. \tag{A4}$$

Furthermore, in view of (16), the second integral of (A1) becomes

$$\int_0^{l} \frac{\partial}{\partial \zeta_1}([u_1{}^0] + i[u_2{}^0])\,\mathrm{d}\zeta_1 = \int_0^{l} \beta\,\mathrm{d}\zeta_1. \tag{A5}$$

From (A3) and (A5), it follows that (A1) holds.

APPENDIX B

Consider the following integral equation:

$$\int_0^l \frac{\psi(t)}{\xi - t}\,\mathrm{d}t - \int_0^l \frac{\psi(t)}{\sqrt{\xi}(\sqrt{\xi}+\sqrt{t})}\,\mathrm{d}t = \frac{A}{\sqrt{\xi}} + B, (0 < \xi < l), \tag{B1}$$

where A and B are constants.

After some manipulation, (B1) yields

$$\int_0^l \frac{\sqrt{t}\psi(t)}{t - \xi}\,\mathrm{d}t = -A - B\sqrt{\xi}, (0 < \xi < l). \tag{B2}$$

Since $\psi(t)$ corresponds to the dislocation density functions and the stress singularity at $\zeta_1 = 0$ is weaker than $1/\zeta_1$, the function $\sqrt{\xi}\psi(\xi)$ vanishes at $\xi = 0$. Therefore, we choose the fundamental function $R(\xi)$ of the solution of (B2) as

$$R(\xi) = \sqrt{\left(\frac{\xi}{l - \xi}\right)}. \tag{B3}$$

Then, the solution of (B2) is given as follows[17]:

$$\sqrt{\xi}\psi(\xi) = \frac{1}{\pi^2} R(\xi) \int_0^l \frac{A + B\sqrt{t}}{R(t)\cdot(t - \xi)}\,\mathrm{d}t, \tag{B4}$$

and, finally,

$$\psi(\xi) = -\frac{A}{\pi}\frac{1}{\sqrt{(l-\xi)}} + \frac{B}{\pi^2}\left\{\log\left|\frac{\sqrt{(l-\xi)}+\sqrt{l}}{\sqrt{(l-\xi)}-\sqrt{l}}\right| - 2\frac{\sqrt{l}}{\sqrt{(l-\xi)}}\right\}, \tag{B5}$$

where we have used

$$\int_0^l \frac{\mathrm{d}t}{R(t)(t-\xi)} = -\pi, \quad \int_0^l \frac{\sqrt{t}}{R(t)(t-\xi)}\,\mathrm{d}t = \sqrt{(l-\xi)}\log\left|\frac{\sqrt{(l-\xi)}+\sqrt{l}}{\sqrt{(l-\xi)}-\sqrt{l}}\right| - 2\sqrt{l}. \tag{B6}$$